How Do We Know That 2+2=4?

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Introduction

This is a survey chapter about issues in the epistemology of elementary arithmetic. Given the title of this volume, it is worth noting right at the outset that the classification of arithmetic as science is itself philosophically debatable, and that this debate overlaps with debates about the epistemology of arithmetic.

It is also important to note that a survey chapter should not be mistaken for a comprehensive, definitive, or unbiased introduction to all that is important about its topic. It is rather an exercise in curation: a selection of material is prepared for display, and the selection process is influenced not only by the author's personal opinions as to what is interesting and/or worthy, but also by various contingencies of her training, and my survey reflects my training in Anglo-American analytic philosophy of mathematics.

Although I'm surveying an area of epistemology, I will classify approaches by metaphysical outlook. The reason for this is that the epistemology and metaphysics of arithmetic are so intimately intertwined that I have generally found it difficult to understand the shape of the epistemological terrain except by reference to the corresponding metaphysical landmarks. For instance, it makes little sense to say that arithmetical knowledge is a kind of "maker's knowledge" unless arithmetic is in some way mind-dependent, or to classify it as a subspecies of logical knowledge unless arithmetical truth is a species of logical truth.

I will be discussing *2+2=4* as an easily-graspable example of an elementary arithmetical truth, our knowledge of which stands in need of philosophical explanation. While some of the surveyed approaches to this explanatory demand proceed by *rejecting* the presumed explanandum—i.e. by denying that *2+2=4* is known (or even true)—for clarity and ease of expression I will proceed as if *2+2=4* is a known truth except when discussing these approaches.

The rest of this chapter proceeds as follows. In the next section, I identify two key challenges for an epistemology of simple arithmetic, and then adduce two constraints on what should count as a successful response. Next, I discuss ways of addressing these challenges, grouped according to their corresponding metaphysical outlook. The subsequent sections survey non-reductive Platonist approaches, look at reductions (often better labelled "identifications"), and consider an array of anti-realist strategies. I conclude with a brief summary, returning to the question of arithmetic's status as science.

Epistemological Challenges

Challenge 1: Abstractness

In this first challenge, we can see immediately how intimately metaphysics is involved in the epistemology of arithmetic. The *locus classicus* for the challenge is 'Benacerraf's dilemma' (Benacerraf 1973). Paul Benacerraf drew attention to two points which together generate the dilemma. The first originates in semantics, and specifically in the fact that arithmetic appears to be *about* objects (numbers). When we talk about arithmetic we appear to refer to numbers, ascribe properties to them, quantify over them, and so on. The second is that if we are to *know* truths about such objects, we must be in some kind of contact with them, and Benacerraf interpreted this as a requirement of *causal* contact. Indeed, he endorsed a causal account of knowledge in general. Such accounts of knowledge are now widely thought to be untenable, but a more sophisticated version of Benacerraf's challenge, developed by Hartry Field (1989), requires only that we be able somehow to explain how we manage to be *reliable* concerning the arithmetical domain, or say how it is that we have mostly true beliefs about them without positing a large-scale coincidence.

Why is this such a big deal? Because of the (prima facie) metaphysics of arithmetic. Arithmetical objects like the numbers 2 and 4, if such things exist, are presumably *abstract objects*. They are not located in space and time; they are not concrete things that we can touch or hear or put under our microscopes; we appear to have no physical contact with them at all. So how do we learn about them? Whether we interpret this question as Benecerraf's demand for a causal account, as Field's demand for an explanation of reliability, or in some other way, it is one of the central challenges in the epistemology of arithmetic.

Challenge 2: A prioricity

This second challenge is more purely epistemological. We seem to know *2+2=4* without *needing* to touch or hear numbers or put them under our microscopes. Indeed, we seem to know it without relying on any kind of empirical evidence at all. That is to say, arithmetical truths appear to be knowable *a priori*. In a nutshell, the second challenge is simply: *what's up with that?* Or: how is *a priori* knowledge possible?

Other kinds of truths also appear to be knowable *a priori*, such as those of logic and set theory. The same challenge is faced by epistemologists in these areas, and there may be some methodological reasons for preferring a unified account of the *a priori* which accommodates all these cases. However, the simplicity and near-universal knowability of basic arithmetical truths like *2+2=4* can make puzzlement about such cases feel all the more urgent, and (as we shall see in a moment) especially challenging.

Constraint i: Applicability

Elementary arithmetic is universally applicable. The breadth of its applicability is even more impressive than that of geometry, which requires some—at least hypothetical—space to describe. By contrast, *2+2=4* is as applicable to poems and headaches as it is to apples and pebbles.

Explaining the applicability of truths about an apparently abstract domain (of numbers) to the physical world (of apples and pebbles) is itself an interesting challenge, but a metaphysical one. For the purposes of epistemology we don't have to provide such an explanation, but we must *leave space* for one. That is to say, whatever story we end up telling about our knowledge of 2+2=4 had better be one that is compatible with the universal applicability of arithmetic.

Constraint ii: Non-specialist knowledge

A significant constraint, albeit one at risk of being overlooked in some accounts (as we see below) is that almost everyone knows that *2+2=4*, and moreover knows it in a way that appears to differ substantively from ordinary empirical knowledge. The challenges of abstractness and *a prioricity* are just as applicable to this non-specialist knowledge as to the knowledge possessed by mathematicians or philosophers. Accounts of arithmetical knowledge which depend on specific expertise—say, on proofs from axioms that *almost nobody* ever considers—cannot meet this constraint without some fancy footwork. Simple fixes are unpromising. One could claim, for example, that non-specialists' knowledge that *2+2=4* is derivative knowledge, reliant on the work done by the mathematical experts. But this is contrary to common sense, which tells us that non-specialists are quite capable of knowing *2+2=4* without the help of mathematicians, as well as that this non-specialist knowledge is *a priori*, which is almost[1] universally regarded as ruling out reliance on someone else's testimony.

The significance of this constraint is part of the reason for this chapter's title. Not only did I deliberately focus on a very widely-known arithmetical truth, I also worded the title in such a way as to invite more explicit reflection on who "we" are: who falls within the scope of the relevant epistemological enquiry? Some theories of arithmetical knowledge may only be applicable to certain kinds of "we"s. This is not invariably a failing; a philosopher might set out to provide an account only of how *expert mathematicians* know *2+2=4*, or only of how *expert philosophers of mathematics* know *2+2=4*. Such accounts do not meet constraint *(ii)* but are not attempting to do so. That said, it is important that the task of accounting for ordinary arithmetical knowledge, the kind possessed by non-specialists, should not fall off the agenda for epistemologists of mathematics considered *en masse*.

Non-Reductive Platonisms

The project of accounting for the knowledge of non-experts has venerable roots in the ancient history of our discipline. In Plato's *Meno*, a classic presentation of a Platonic epistemology of mathematics, an uneducated slave-boy's knowledge of geometry is used as a focal example of the kind of phenomenon in need of philosophical explanation. To address this need, Plato's Socrates offers a theory of *a priori* knowledge which appeals to the broader Platonic doctrine of recollection. According to this doctrine (developed further in other dialogues, including the *Republic*, *Phaedo*, and *Phaedrus*), one is able to know certain things *a priori* because one is *remembering* things learned as a disembodied soul, prior to one's (current) incarnation. Souls are in direct contact with idealized abstract objects—the Forms—and can, when prompted, recall information about them. Taking arithmetical objects such as the natural numbers to be among the Forms, known prior to birth and recollected when prompted by teachers or experience, supplies a simple account of how we—that is, any or all of us—can know *a priori* that *2+2=4*.

While Plato's mythology of reincarnation is no longer regarded as the basis for a promising account of *a priori* mathematical knowledge, contemporary views are often labelled forms of *Platonism* if they posit mind-independent, abstract mathematical objects. *Non-reductive* Platonisms moreover make no attempt to identify these objects as being of some (relatively) philosophically untroubling variety, but rather embrace their distinctive abstract status. Such Platonisms cleave strongly to the first half of Benacerraf's dilemma: arithmetic *appears* to be about distinctive abstract arithmetical objects because it *is* about such objects.

These views thus confront the challenge of the dilemma's second half in its purest form: how can we account for knowledge of such objects? One famous (or infamous) answer is inspired by philosopher-mathematician Kurt Gödel's remark that "despite their remoteness from sense-experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true." He continued: "I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical

[1] But see Burge 1993 (and elsewhere).

intuition, than in sense-perception" (Gödel 1947, pp. 483-4). While Gödel's primary interest here was in set theory, one might easily extend or adjust the view to postulate "something like a perception" of numbers.

Positing such a faculty (often labelled 'intuition' or 'rational intuition') is not a popular contemporary option, largely because it is felt to be in tension with *naturalism*. Naturalism is generally seen as demanding a scientific worldview grounded in empirical evidence; a faculty of intuition sits uneasily with this, both because it would itself *be* a non-empirical epistemic source, and because there is a dearth of empirical evidence for its existence.

Another non-reductive option is known as 'plenitudinous' Platonism (see Balaguer 1998). According to this view, all consistent mathematical theories are true, and all of their objects exist. So *any* mathematical proposition one believes—as long as it is not inconsistent—is true. Thus, Field's demand for an explanation of the reliability with which we believe truly in the mathematical domain is supposedly put to rest: the accomplishment is trivial, hence there's nothing to explain.[2]

To the extent that this feels unsatisfying, that feeling may flag a problem with Field's move to reposition Benacerraf's challenge as a demand for such an explanation. Recall that what Benacerraf originally sought was a viable pairing of accounts, one semantic and one epistemological, to deal with propositions like *2+2=4*. Plenitudinous Platonism, at least in its purest form as an ontological theory, supplies neither. Another (related) kind of objection is that it is never an entirely trivial accomplishment to *refer to* or to *know about* some objects; merely postulating the existence of *lots* of objects does not change that.

Non-reductive Platonisms may be attractive insofar as they hold out the prospect of respecting both the abstractness and the *a prioricity* of arithmetic. They also tend to be well-positioned to account for all kinds of arithmetical knowledge, expert and non-expert alike, in ways that may differ in degree but do not require entirely different epistemologies. Their problems, in my opinion, tend to arise in connection with the applicability constraint. It is difficult to explain the applicability of abstract mathematical objects to the physical world, and my diagnosis is that it is precisely this explanatory gap which underwrites widespread suspicion that any genuine attempt to explain how such flesh-and-blood concrete creatures as ourselves could have *knowledge* of abstracta will be "spooky" and insufficiently naturalistic.

In the twentieth century, Willard van Orman Quine made a sustained effort to marry non-reductive Platonism (particularly about sets) with a naturalistic and empiricist epistemology for mathematics. The most influential statement of this project's outlines can be found in Quine (1951), and the core ideas are developed in the work of many later philosophers (see e.g. Colyvan 2001; Devitt 2005). Quine proposed a form of epistemological *holism*, according to which it is not individual beliefs or hypotheses that are tested against experience, but rather entire worldviews as a package deal. When empirical confirmation is received, it accrues to everything in the package at once, including mathematical beliefs. According to such a view, our knowledge that *2+2=4* is not in fact *a priori*, but a (long-established) element of our best empirically confirmed overall theory. As has been emphasized by Devitt in particular, the holist attempts to *explain away* the appearance of *a prioricity*: to defuse *challenge (ii)*, rather than meet it head-on.

The Quinean approach can, however, be understood as a direct attempt to accommodate the abstractness of mathematics — *challenge (i)* — in tandem with the applicability constraint. It's not an *accident* that we choose the mathematics that is applicable to the world as we experience it:

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[2] Avoiding inconsistency is non-trivial in some cases, but when our attention is focused on propositions like *2+2=4* we might reasonably waive that concern.

mathematics is deployed in our theories precisely to be so applicable, to help us explain and predict. That theories about abstracta should be helpful in these ways may be surprising to philosophers, but they nevertheless appear to be (not merely helpful but) *indispensable* for best science. Whether this truly meets the challenge of applicability is up for debate, however. What this approach explains is *why certain applicable mathematical theories are the ones we adopt*, not *why those particular ones are applicable in the first place.*

The holistic approach may also face questions as to who are the relevant 'we'. Most people do not depend on much advanced mathematics in their understanding of the world, but if the relevant 'we' is set as (say) some expert community of scientists, the question remains of what *their* situation has to do with everyday, non-expert knowledge of *2+2=4*.

Reductions/Identifications

If one wants to believe in an arithmetical reality—a mind-independent realm for propositions like *2+2=4* to be about—without committing to any arithmetical objects that feel too "spooky" or strange for a respectable naturalistic ontology, one can try arguing that arithmetical reality is identical with some realm of reality that is less spooky-sounding, at least on the face of it, and/or some realm of reality already accepted as *bona fide*.

One might, for example, identify particular arithmetical objects (like the number 4) with naturalistically respectable objects. This has the advantage of proving a relatively simple semantics for *2+2=4*, like non-reductive Platonisms. Another option is to identify arithmetical reality more broadly with some unspooky realm of reality, without making specific object-identity claims. This option leaves more semantic questions up in the air.

Such identification strategies are often called *reduction* strategies, the idea being that arithmetic is *reduced* to a domain of reality that is already acceptable, thus avoiding the 'inflated' ontology of a non-reductive Platonism. This terminology is in some ways unfortunate, since 'reduction' carries connotations of asymmetry that 'identification' does not. If the posited identities really do obtain, such connotations may be misleading (since identity is symmetric).

One well-known (but not popular) identification strategy is logicism, which in broad-brush terms is the view that arithmetic is part of logic. Like many of the views in this chapter, logicism comes in more and less general varieties: one can be a logicist about all of mathematics, or about specific subdisciplines such as arithmetic. Different areas of mathematics offer different prospects and problems for logicist identification. Arithmetic is, however, the area whose susceptibility to a logicist treatment has been most thoroughly investigated. The classic attempt to establish an identification is Gottlob Frege's two-volume work *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*) published in 1893 and 1903. This attempt was doomed when Frege's proposed Basic Law 5 turned out to be inconsistent—as established by Bertrand Russell, it leads to a paradox.[3]

When it comes to epistemology, a logicist has no single, obviously-preferable option, but can at least argue that what appeared to be two sources of epistemological puzzlement are really one. If arithmetic is a branch of logic, then arithmetical knowledge is a species of logical knowledge, and our preferred epistemology of logic becomes an epistemology for arithmetic also. Logicism is also relatively well-placed to make space for the abstractness and *a priori* knowability of arithmetic, at least on the assumption that logic shares these features. With respect to the

[3] This is "Russell's Paradox," the famous problem that arises when we consider a set whose members are *all the sets that are not members of themselves*.

applicability of arithmetic, the logicist again starts on firm ground: logic is applicable to reasoning about absolutely anything.[4] However, the ubiquity of non-specialist arithmetical knowledge risks throwing a spanner in logicist works. Frege's logicist derivations of the standard axioms for arithmetic (known as the Peano Axioms) are extremely specialized. Almost nobody in the world has learned how to perform these derivations, or anything remotely like them, and yet almost everybody in the world knows that *2+2=4*.

A second well-known identification strategy is arithmetical *structuralism*, which attempts to establish that arithmetic is the study of certain structures, hoping thus to avoid the non-reductive Platonist's commitment to spooky objects. One formulation is due to Stewart Shapiro (1997), who outlined what is sometimes called an *ante rem* variety of structuralism (to distinguish it from *in rebus* structuralism, of which more in the next section). According to a Shapiro-style structuralist, structures are characterized by structural relations. These structures exist independently of any objects which exemplify their defining relations (hence the label 'ante rem'). Knowing that *2+2=4* is thus knowing something about the natural number structure; in particular, it is knowing something about the relations that define certain positions (*2* and *4*) within that structure.

Structures of this kind are still abstracta, and questions remain as to whether they are a sufficiently comprehensible or unspooky kind of abstracta. This form of structuralism is among the most promising of the views surveyed in this chapter with respect to the universal *applicability* of arithmetic, since it makes arithmetic the study of structural relations which absolutely any kind of object can instantiate. (It also tidily explains what application actually consists in, i.e. instantiation.) Non-specialist knowledge is also accommodated, on the plausible assumption that a structure is the kind of thing one can know a little or a lot about, and about which one might learn in various (formal and informal) ways. The view may also appear to fare well in accommodating the abstractness of arithmetic, since it renders arithmetic as the study of abstracta; however, an effective response to Benacerraf's challenge requires in addition some explanation of how this abstract domain is known to us, and (apparently) known *a priori*. One attempt to develop an epistemology for arithmetic that sits well with structuralism of this bent is found in Jenkins (2008),[5] where I discuss the hypothesis that aspects of the world's arithmetical structure impacts us through sense experience, and thereby epistemically "grounds" our most basic arithmetical concepts, from which we may then recover information about that structure.

Anti-Realisms, type a (no-truths) and type b (no-objects)

Further alternatives to non-reductive Platonism can be bundled together under the umbrella of *anti-realism* (although that label admits of such broad and variable usage that its application to any of these positions is not always particularly illuminating until further clarification is appended). Broadly speaking, there are three main ways to be an anti-realist about arithmetic:

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[4] See MacFarlane 2015, §4 for a helpful discussion of the notion of *topic neutrality*, which untangles a potential complication here. One kind of topic-neutrality *is* universal applicability; both logic and arithmetic are topic-neutral in this sense. In the other sense, to be topic-neutral is to contain no expression that discriminates between particular objects. Since arithmetic does appear to contain some such expressions (e.g. 'is the number 4'), it is not straightforwardly classifiable as topic-neutral in the second sense. But if it turns out to have been only logic all along, it may be possible to argue that arithmetic is topic-neutral in both senses.

[5] See especially chapters 4 and 5.

a. Deny that arithmetical propositions such as *2+2=4* are true.
b. Allow that they are true, but deny that they are about objects (such as numbers).
c. Allow that they are true (and perhaps about objects), but deny that their truth is mind-independent.

Option a is perhaps the most counterintuitive. While it may sound extreme, however, it promises a get-out-of-jail-free card to the metaphysician troubled by abstracta, and derivatively (albeit more relevantly for our purposes) to the epistemologist troubled by Benacerraf-style worries. If arithmetic is not a domain of truths, then there are no mysterious arithmetical objects and no mysterious arithmetical knowledge to account for.

Two principal versions of this *no-truths* form of arithmetical anti-realism are fictionalism and formalism. According to the fictionalist, arithmetical propositions like *2+2=4* are strictly speaking false, although it might be very useful to treat them as if they were true for certain purposes. In the ontology room, the fictionalist says, we must say that they are false and hence do not commit us to peculiar objects like the number 4.[6] Field is known for defending a form of fictionalism in his 1980 book *Science Without Numbers*, where he argued that the use of arithmetic in scientific applications does not commit us to the truth of the arithmetical propositions so applied.

While the formalist agrees with the fictionalist that arithmetical propositions like *2+2=4* are not true, the formalist says they are not false either, but (strictly speaking) meaningless.[7] The activity we call 'arithmetic' is an activity in which we manipulate formal symbols like '2' and '4', and while we have rules for such manipulations which permit the string '2+2=4' and forbid the string '2+2=5' , this is not because of what they *mean*. Neither string, in fact, has any semantic content.

Fictionalists and formalists alike sidestep the difficulties associated with accounting for arithmetic's *a priori* knowability and the abstractness of its subject matter. They can comfortably accommodate the role of non-specialists, too, as inexpert participants in the (fictional or formal) activity of arithmetic. However, such strategies come at a high cost when it comes to the applicability of arithmetic. In a discussion of formalism, in §91 of volume II of the *Grundgesetze*, Frege stated that 'it is applicability alone which elevates arithmetic above a game to the rank of a science' (1893/2013, p. 100). The equivalent issue arises for fictionalism, where it becomes a version of the Putnam-Boyd *no-miracles* argument against anti-realism in the philosophy of science:[8] if the theory isn't (by and large) true, why does it work so well?

Option b is *no-objects* anti-realism about arithmetic. The term *nominalism* is sometimes used for this kind of view (derivatively upon its use to describe the rejection of universals and/or abstract objects in general). Fictionalism and formalism deliver nominalism fairly directly—indeed, the full title of Field's fictionalist 1980 is *Science Without Numbers: A Defence of Nominalism*.

In mathematical contexts, the *no-miracles* argument has been sharpened into what is known as the Quine-Putnam *indispensability* argument against nominalism.[9] It runs roughly as follows: propositions quantifying over numbers are indispensable elements of our best scientific

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[6] Fictionalism about arithmetic admits of many possible permutations with respect to the details of the view. For a good overview, see Balaguer (2015).
[7] Early versions of formalism, made famous as the subject of Frege's critique in his *Grundgesetze*, were unclear on this point. See section 2 of Weir 2015 for a helpful historical discussion.
[8] Developed in e.g. Putnam 1975 and Boyd 1989.
[9] See Colyvan 2001 for a thorough treatment.

theories, and (applying Quine's criterion of ontological commitment) this means that accepting those theories commits us to the existence of numbers. We accept the theories, so we must accept the numbers.

It is this kind of argument to which Field was responding in his 1980 book, attempting to show that scientific theories can be viably reformulated to avoid quantification over numbers. His response is limited in scope (he only addressed part of Newtonian gravitational theory), and it is controversial whether it succeeds even within its limits—for example, one may dispute whether his convoluted nominalistic proposal is a viable contender for a *good* (never mind our best) scientific theory, given that simplicity is generally considered an important theoretical virtue, and whether Field's proposal sneaks in abstract number-substitutes in the guise of what he called 'space-time regions'.

But one may also be a nominalist for less drastic reasons than being a fictionalist or formalist. Some forms of structuralism are best classified as *type b* anti-realisms (but *not type a*), especially those that reject from their ontologies not only numbers but also structures. These are known as *in rebus* structuralisms because they hold that arithmetic is the study of certain kinds of structural arrangements in which concreta stand, but that no abstract structures exist beyond, or in addition to, the instantiating concrete things. This type of view, however, renders mathematical truth hostage to the factual question of how many concreta exist.[10]

Nominalist structuralisms duck the challenge of explaining how we can have knowledge concerning a realm of abstract mathematical objects by denying that there are any such objects. Such views also inherit some of the advantages of Platonist *ante rem* structuralism when it comes to accounting for non-expert knowledge, and for the applicability of arithmetic. It should be noted, however, that the modalized versions are less well placed in both regards, as they may end up recommitting to abstracta in their modal ontologies, and face the additional task of explaining why merely possible structures are applicable to the actual world. When it comes to accounting for the apparent *a prioricity* of arithmetical knowledge, however, nominalist structuralists have significant work to do.

Perhaps the appearance of *a prioricity* can be explained away. If this is the task, it is one shared by a quite different form of nominalism, less metaphysically sophisticated than that of the *in rebus* structuralists, but important enough to discuss here. In his 1843 work *A System of Logic*, J.S. Mill famously denied that arithmetical knowledge is *a priori*. He maintained that arithmetic is simply a branch of the empirical, *a posteriori* study of nature, and that arithmetical theorems are among the most general laws of nature. In effect, *2+2=4* states that any two objects added to any other two objects makes four objects. But "[a]ll numbers must be numbers of something: there are no such things as numbers in the abstract" (1843, Vol. I, Book II, Chapter VI, §2).

This Millian form of empiricism takes non-specialist knowledge and the applicability of simple arithmetic in its stride, but faces difficulties accounting for the application of arithmetic to extremely large numbers of things that we have never experienced (and that may not even exist). This problem also promises to recur in spades when one moves from arithmetic to consider higher mathematics. It is also important to note in general that directly rejecting the abstractness and *a prioricity* of arithmetic is no epistemological get-out-of-jail-free card, since finding an adequate empirical basis for *all* knowledge of arithmetic (not just knowledge of elementary sums)

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[10] And because this strikes many as counter-intuitive, modalized alternatives have been explored, for example in Geoffrey Hellman's (1989) *Mathematics Without Numbers*. On this approach, arithmetic becomes the study of *possible* structures.

is a huge task. (Moves to less simplistic empiricist views, such as Quinean holism, are motivated by precisely such considerations.)

Anti-Realisms, type c (mind-dependence)

The third and final kind of anti-realism to be surveyed here is *type c*, or mind-dependence, anti-realism. Several well-known—but very different—positions come under this heading.

The first is Kant's conception of arithmetic, as put forward in his 1781 *Critique of Pure Reason*. Kant says that arithmetic is the form of our 'temporal intuition'. While Kant exegesis is a fraught business, one way of unpacking this has Kant saying that the way we (human agents) experience the world is temporally structured, while the world as it exists in itself—that is, outside of experience—cannot be assumed to share this temporal structure. Time is thus an aspect of *our* ways of thinking, feeling, experiencing and understanding. It is temporal structure so understood that is—both metaphysically and epistemologically speaking—the *basis* of arithmetic, according to Kant.

This Kantian approach has been a strong contender in the epistemology of arithmetic ever since, and for good reason, as can be seen by noting its potential for accommodating the criteria that structure this chapter. The applicability of arithmetic is built right into the account: arithmetic is the study of the (temporal) structure of experience, so naturally it is applicable to the world as we experience it. Non-specialist knowledge is also straightforwardly accounted for: we all share this temporal form of intuition. Next, arithmetical knowledge is positioned as a form of self-knowledge, since arithmetic is an aspect of our own contribution to the world of experience, and this is widely held to be helpful in accounting for its *a prioricity*: one looks *inward* rather than outward to learn about arithmetic. (However, one might have reservations about this, as certain kinds of self-knowledge are hard to come by, and cannot be secured through introspection.) Whether the proposal must *explain* or *explain away* knowledge of abstract objects such as numbers will depend on how one spells out the details of the Kantian view; its emphasis on time as *structuring* our experience might be developed into something resembling a Platonist or a nominalist form of structuralism.

Wittgenstein (especially in his *Remarks on the Foundations of Mathematics*, a collection of notes published in 1956) proposes a different, and in some ways more radical, kind of mind-dependence. Wittgenstein argues that all of mathematics is created (or invented) by us. In keeping with his broader interest in the nature of rules and rule-following, he treats arithmetic (and mathematics in general) as a domain of normative rules, such as the rule that says when adding 2 and 2 you should give the answer 4. But such rules, he argues, are created by us in the process of practicing arithmetic—that is to say, by calculating, counting, and so on, we generate arithmetical truths. This is a (particularly stark) form of *constructivism* in the metaphysics of arithmetic, which has historical precedents in the work of L. E. J. Brouwer.[11] Brouwer is also known as an *intuitionist* because he held that it is the possibility of constructing a proof in the mind—in one's 'intuition'[12]—that renders a mathematical proposition true.

As far as epistemology goes, constructivist proposals face a suite of advantages and disadvantages. On the one hand, arithmetical knowledge does not involve access to a realm of

[11] See Brouwer 1981 (a collection of material for lectures Brouwer delivered between 1946 and 1951).

[12] Cautionary note: the English word 'intuition' is used in translations of both Brouwer and Kant, but it should not necessarily be assumed to mean exactly the same in each case.

spooky abstract objects. Whether there are arithmetical objects at all (and whether—and in what sense—they are abstract) depends on the details of one's constructivist metaphysics, but the realm of arithmetic is at least restricted to the realm of the constructed—that is, constructed *by someone*—and is to that extent all "within reach." The *a prioricity* of arithmetical knowledge might be approached in a somewhat Kantian spirit: if arithmetical knowledge is *maker's* knowledge, this will differentiate it in certain ways from the knowledge of *discovered* facts. The applicability of arithmetic needs addressing but need not be an insurmountable challenge. (Why would an invented or constructed arithmetic apply tidily to the world? Perhaps because it was built specifically for that purpose.)

Non-specialist knowledge is more awkward, however, at least for the intuitionist, since most non-mathematicians' knowledge of simple arithmetical truths has little to do with *proof* (or construction) in anything like a mathematician's sense. Whether or not this is an issue for Wittgenstein's constructivism is hard to pin down, due to his often vague style of writing (and perhaps, depending on one's exegesis, his changes of mind). He tends to use first-person plural language ('we', 'our') without specifying who is included.

A third category of mind-dependence anti-realism is worth canvassing briefly. This is the class of views on which true arithmetical statements are *analytic*: made true by the meanings of words, or by relations between concepts. This approach finds a *locus classicus* in the work of the Logical Empiricists, and particularly Carnap, who argues that we adopt a framework of arithmetical concepts for pragmatic reasons: that is, because such a framework is useful in application to the physical world (see Carnap 1950). The question of whether numbers exist can then be asked as a question *internal* to this framework, in which case the answer is trivially "yes," or it can be asked as an *external* question, in which case it is empty (because talk of numbers only makes sense within the arithmetical framework).

On this view, our *a priori* knowledge of arithmetic is accounted for as knowledge concerning the framework of arithmetical concepts that we have adopted. The applicability of such knowledge is built into the view: the framework has been adopted *because* it is useful. Non-specialist knowledge is also relatively unproblematic, insofar as anyone who is using the arithmetical framework might be supposed to know some elementary facts about the relations between its constituent concepts. The strangeness of knowledge concerning abstract arithmetical objects is purportedly defused: it is a trivial matter to answer *internal* questions about such objects, and those are the only kinds of questions one can sensibly ask about them. (Precisely this kind of *deflation* of arithmetic's ontology, however, may prove a sticking point for those struck by the apparent robustness of arithmetic.)

A more sophisticated descendent of the analyticity view is *neo-Fregeanism*, initially developed by Crispin Wright and Bob Hale (see Wright 1983; Hale and Wright 2001). This work secures a derivation of the standard Peano Axioms for arithmetic from a single premise, known as Hume's Principle, which is then positioned as analytic (or an implicit definition) of *number*. This, in some respects, loops us back around to logicism (and, indeed, this neo-Fregean view sometimes also goes by the name *neo-logicism*): for, like logicists, neo-Fregeans maintain that all that is required to ground arithmetic are logic and definitions.

A priori arithmetical knowledge is accounted for by neo-Fregeans as knowledge of analytic or definitional matters (though the *a prioricity* of logic still requires its own explanation). Because this is a crucial point, much subsequent debate has centered on whether or not the premise known as Hume's Principle *is* in fact analytic in the sense required to deliver these epistemological results.

It is also worth noting that neo-Fregeanism inherits some potentially troubling features of its logicist ancestors, including apparent inapplicability to non-specialist knowledge of arithmetic (which does not seem to proceed from knowledge of anything like Hume's Principle, or the derivations therefrom of the Peano Axioms). Neo-Fregeanism also supplies no particularly obvious explanation for the applicability of arithmetical knowledge to the physical world.

Conclusions

The question of how we know that *2+2=4* remains hotly debated in contemporary epistemology of mathematics. As the foregoing discussion evinces, the answers preferred by different philosophical camps depend heavily on what kind of truth (if any) they take *2+2=4* to be. In particular, metaphysical divisions between non-reductive Platonists, reductionists, and anti-realists of various stripes are reflected in very different approaches to arithmetical epistemology.

In a similar manner, the extent to which philosophers are inclined to classify arithmetic as a *science* tends to be reflected in a corresponding disinclination to emphasize its apparent epistemological differences from the (paradigmatic, or stereotypical) natural sciences. In particular, mathematical naturalists who (following Quine) classify arithmetic firmly as part of science tend correspondingly to downplay or deny the *a prioricity* of arithmetic, emphasizing instead its dependence on empirical confirmation.

Like the question of how we know *2+2=4*, the question of whether arithmetic is a science or not remains unsettled, lacking even an emergent consensus. To an extent, however, this latter question is one about the contingencies of disciplinary borders and categories, and as such might be most helpfully addressed by sociologists, anthropologists, and historians of academia.

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